2018-2019 Yılı, MAT 311 Nümerik Analize Giriş Bütünleme Sınavı

Numara: İsim-Soyisim:

SORULAR

1. $(x-1)^2 = 0$ denklemini Newton-Raphson yöntemi ile çözünüz.

2. $\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= -1 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$ denklem sistemini LU yöntemi ile çözünüz.

3. $f(x) = e^x$ için $R_{1,1}(x)$ Pade yaklaşımını elde ediniz.

4. Lagrange polinomunu kullanarak $O(h^2)$ mertebeden $f''(x_0) \approx \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$ geri fark formülünü elde ediniz.

5.

i	0	1	2
x_i	0	2	3
y_i	7	11	28

noktalarını kullanarak Newton yöntemi ile 2. dereceden polinom oluşturup $x = 1$ noktasında y değerini hesaplayınız.

CEVAPLAR

1) $P_k = P_{k-1} - \dfrac{f(P_{k-1})}{f'(P_{k-1})}$, $k = 1, 2, 3, \ldots$ $\quad p_0 \in [p - \delta, p + \delta]$

$f(x) = (x-1)^2 \to f(P_{k-1}) = (P_{k-1} - 1)^2, \; f'(P_{k-1}) = 2(P_{k-1} - 1)$

$P_k = P_{k-1} - \dfrac{(P_{k-1} - 1)^2}{2(P_{k-1} - 1)} = P_{k-1} - \dfrac{P_{k-1} - 1}{2} = \dfrac{P_{k-1}}{2} + \dfrac{1}{2}$, $k = 1, 2, 3, \ldots$

$k = 1$ için $P_1 = \dfrac{P_0}{2} + \dfrac{1}{2}$

$k = 2$ için $P_2 = \dfrac{P_1}{2} + \dfrac{1}{2} = \dfrac{P_0}{2^2} + \dfrac{1}{2^2} + \dfrac{1}{2}$

$k = 3$ için $P_3 = \dfrac{P_2}{2} + \dfrac{1}{2} = \dfrac{P_0}{2^3} + \dfrac{1}{2^3} + \dfrac{1}{2^2} + \dfrac{1}{2}$

Tümevarım yönteminden

$$P_k = \frac{P_0}{2^k} + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k}$$ bulunur. $P_0 = 0$ için

$$P_k = 0 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k}$$

$$P = \lim_{k \to \infty} P_k = \lim_{k \to \infty} \left(0 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k}\right) = \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = 1$$

bulunur.

2) $$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{U}$$

$u_{11} = 1$, $m_{21} = 2$, $m_{31} = 1$, $u_{12} = -1$

$-2 + u_{22} = 1 \to u_{22} = 3$, $u_{13} = 1$

$2 + u_{23} = -1 \to u_{23} = -3$

$-1 + 3m_{32} = 1 \to m_{32} = \frac{2}{3}$

$1 + (-2) + u_{33} = 1 \to u_{33} = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$LUx = B \rightarrow Ux = Y \rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} y_1 = 1 \\ y_2 = -3 \\ y_3 = 2 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 1 \end{matrix} \rightarrow X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3) $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \dots$

$$R_{1,1}(x) = \frac{b_0 + b_1 x}{1 + a_1 x} \quad \text{dir.}$$

Buradan

$$(1 + a_1 x)\left(1 + x + \frac{x^2}{2} + \dots\right) - b_0 - b_1 x = O(x^3)$$

x^0: $1 - b_0 = 0 \rightarrow b_0 = 1$

x : $a_1 + 1 - b_1 = 0 \rightarrow b_1 = \frac{1}{2}$

x^2: $\frac{1}{2} + a_1 = 0 \rightarrow a_1 = -\frac{1}{2}$

bulunur.

Bu nedenle

$$R_{1,1}(x) = \frac{1+\frac{x}{2}}{1-\frac{x}{2}} \qquad \text{bulunur}$$

4) $(x_0, f_0), (x_{-1}, f_{-1}), (x_{-2}, f_{-2}), (x_3, f_{-3})$ noktalarından geçen $f(x)$ fonksiyonu için Lagrange polinomu,

$$f(x) \approx f_0 \frac{(x-x_{-1})(x-x_{-2})(x-x_{-3})}{(x_0-x_{-1})(x_0-x_{-2})(x_0-x_{-3})} + f_{-1} \frac{(x-x_0)(x-x_{-2})(x-x_{-3})}{(x_{-1}-x_0)(x_{-1}-x_{-2})(x_{-1}-x_{-3})}$$

$$+ f_{-2} \frac{(x-x_0)(x-x_{-1})(x-x_{-3})}{(x_{-2}-x_0)(x_{-2}-x_{-1})(x_{-2}-x_{-3})} + f_{-3} \frac{(x-x_0)(x-x_{-1})(x-x_{-2})}{(x_{-3}-x_0)(x_{-3}-x_{-1})(x_{-3}-x_{-2})}$$

dir. İki kere türevini alırsak

$$f''(x) \approx f_0 \frac{2\left[(x-x_{-1})+(x-x_{-2})+(x-x_{-3})\right]}{(x_0-x_{-1})(x_0-x_{-2})(x_0-x_{-3})} + f_{-1} \frac{2\left[(x-x_0)+(x-x_{-2})+(x-x_{-3})\right]}{(x_{-1}-x_0)(x_{-1}-x_{-2})(x_{-1}-x_{-3})}$$

$$+ f_{-2} \frac{2\left[(x-x_0)+(x-x_{-1})+(x-x_{-3})\right]}{(x_{-2}-x_0)(x_{-2}-x_{-1})(x_{-2}-x_{-3})} + f_{-3} \frac{2\left[(x-x_0)+(x-x_{-1})+(x-x_{-2})\right]}{(x_{-3}-x_0)(x_{-3}-x_{-1})(x_{-3}-x_{-2})}$$

elde edilir. $x = x_0$ ve $x_i - x_j = (i-j)h$ olduğundan

$$f''(x_0) \approx f_0 \frac{2\left[(x_0-x_{-1})+(x_0-x_{-2})+(x_0-x_{-3})\right]}{(x_0-x_{-1})(x_0-x_{-2})(x_0-x_{-3})} + f_{-1} \frac{2\left[(x_0-x_0)+(x_0-x_{-2})+(x_0-x_{-3})\right]}{(x_{-1}-x_0)(x_{-1}-x_{-2})(x_{-1}-x_{-3})}$$

$$+ f_{-2} \frac{2\left[(x_0-x_0)+(x_0-x_{-1})+(x_0-x_{-3})\right]}{(x_{-2}-x_0)(x_{-2}-x_{-1})(x_{-2}-x_{-3})} + f_{-3} \frac{2\left[(x_0-x_0)+(x_0-x_{-1})+(x_0-x_{-2})\right]}{(x_{-3}-x_0)(x_{-3}-x_{-1})(x_{-3}-x_{-2})}$$

$$= f_0 \frac{2(h+2h+3h)}{h.2h.3h} + f_{-1} \frac{2(2h+3h)}{-h.h.2h} + f_{-2} \frac{2(h+3h)}{-2h.(-h).h} + f_{-3} \frac{2.(h+2h)}{(-3h)(-2h).(-h)}$$

$$= f_0 \frac{12h}{6h^3} + f_{-1} \frac{10h}{-2h^3} + f_{-2} \frac{8h}{2h^3} + f_{-3} \frac{6h}{-6h^3}$$

$$= \frac{12h f_0 - 30h f_{-1} + 24h f_{-2} - 6h f_{-3}}{6h^3}$$

$$= \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$$

5) $P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

$a_0 = f[x_0] = f(x_0)$

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\dfrac{f[x_2] - f[x_1]}{x_2 - x_1} - \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$x_0 = 0$, $f(x_0) = y_0 = 7$

$x_1 = 2$, $f(x_1) = y_1 = 11$

$x_2 = 3$, $f(x_2) = y_2 = 28$

$$P_2(x) = 7 + \frac{11-7}{2-0}(x-0) + \left(\frac{\frac{28-11}{3-2} - \frac{11-7}{2-0}}{3-0} \right)(x-0)(x-2$$

$$= 7 + 2x + 5x^2 - 10x$$

$$= 5x^2 - 8x + 7$$

bulunur. Buradan

$$P_3(1) = 5.1^2 - 8.1 + 7 = 4$$

olarak elde edilir.